Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON THE CLASSIFICATION OF PLANE ALGEBRAIC CURVES POSSESSING FOURFOLD SYMMETRY WITH RESPECT TO A POINT*

BY R. D. CARMICHAEL

A PLANE figure is said to be symmetrical with respect to a point, called the center of symmetry, if it coincides with its original position after being turned through 180° in its plane about that center. This we may call twofold symmetry.

Generally, we shall say that a plane figure has r-fold symmetry with respect to a point, called the center of symmetry, if it coincides with its original position after being turned through $\frac{360^\circ}{r}$ in its plane about that center.

The object of this paper is the classification of irreducible plane algebraic curves possessing fourfold symmetry with respect to a point. The work will be carried out by showing that (excluding the case of curves consisting solely of isolated points) there are no such curves of odd degree, and that there are two classes of such curves of even degree.

If the origin of rectangular Cartesian coordinates is taken at the center of symmetry, and if (α, β) is a point on the curve, then also is $(-\alpha, -\beta)$, as may be readily seen from the above definition of fourfold symmetry. Then if these two sets of values are put for x, y in the equation of the curve there will result two equations which differ only in respect to the signs of the terms of odd degree in α, β, all such signs being different in the two cases. Take the sum of these two equations if their degree is odd; and their difference if their degree is even: in each case the resulting equation will be one of degree $\leqq n-1$, n being the degree of the original equation. This is inconsistent with the assumption of a locus of the nth order unless the degree of this last equation is zero; that is, unless the equation vanishes. For, suppose the contrary; we should then have a locus of order less than n containing all the points (α, β) of an nth degree locus; and this is impossible since the curve is supposed irreducible. This leads readily to the following preliminary theorem:

---

* Read before the American Mathematical Society, September 5, 1907.

The equation in rectangular Cartesian coordinates of a plane algebraic curve of order n, possessing fourfold symmetry with respect to a point taken as the origin, has only terms of odd degree in x, y when n is odd, and only terms of even degree in x, y when n is even.

This equation may be written

$$\sum a_{ts} x^t y^s = 0, \tag{1}$$

where t and s each ranges from 0 to n subject to the condition

$$t + s = 1, 3, 5, \cdots, n \quad \text{when } n \text{ is odd,}$$
$$t + s = 0, 2, 4, \cdots, n \quad \text{when } n \text{ is even,}$$

and where a_{ts} are real constants so far taken independent of each other. We proceed now to find certain relations which exist among them.

For this purpose it will be found convenient to transform the equation to polar coordinates, the origin remaining the same and the angle θ being measured from the x-axis. The equation then is

$$\sum a_{ts} \cos^t\theta \sin^s\theta\, \rho^{t+s} = 0, \tag{2}$$

where t and s are subject to the same conditions as before. Now, if $90° + \theta$ is put for θ, equation (2) becomes

$$\sum (-1)^t a_{ts} \sin^t\theta \cos^s\theta\, \rho^{t+s} = 0. \tag{3}$$

In order that the loci should possess the defined fourfold symmetry with respect to the origin it is both necessary and sufficient that the same values of ρ satisfy both equations (2) and (3) when θ has any fixed value. Therefore the coefficients of like powers of ρ can differ only by some constant factor m. Therefore we must have

$$\sum a_{ts} \cos^t\theta \sin^s\theta = m \sum (-1)^t a_{ts} \sin^t\theta \cos^s\theta, \tag{4}$$

where the summation is extended to all permissible values of t and s such that $t + s$ has any fixed value.

We assume that the locus does not consist entirely of isolated points.

Then there are an infinite number of values of θ for which $\cos\theta$ is not zero in every case for which $n > 1$; and evidently when $n = 1$ the fourfold symmetry does not exist. Then transposing in (4) and dividing by $\cos^{t+s}\theta$, we have

$$(5) \qquad \sum a_{ts}\tan^s\theta - m\sum(-1)^t a_{ts}\tan^t\theta = 0,$$

$t + s$ being fixed in value. Now, (5) is an algebraic equation in $\tan\theta$ of degree not exceeding n; but it must be satisfied by an infinite number of values of θ. Therefore its first member vanishes identically; and hence the coefficient of any power of $\tan\theta$ must vanish. That is, we must have

$$(6) \qquad a_{st} - m(-1)^t a_{ts} = 0.$$

Interchanging t and s we arrive at the equation

$$a_{ts} - m(-1)^s a_{st} = 0;$$

whence

$$m(-1)^t a_{ts} - m^2(-1)^{t+s} a_{st} = 0.$$

Combining with (6), we have

$$a_{st} - m^2(-1)^{t+s} a_{st} = 0;$$

or

$$1 - m^2(-1)^{t+s} = 0, \qquad \text{or} \qquad a_{st} = a_{ts} = 0.$$

In the latter case, $a_{st} = a_{ts}$, equation (1) vanishes. In the former case, if n is odd so is $t + s$; and therefore we have

$$m = \pm\sqrt{-1};$$

that is, two real coefficients differ by an imaginary factor, which is impossible. Therefore we have the following theorem:

Excluding curves consisting of isolated points only, there exist no loci of odd order possessing fourfold symmetry with respect to a point.

If n is even so is $t + s$; then we have

$$m = +1, \quad -1.$$

Hence from (6) it follows readily that there are to be considered the two cases

$$(7) \qquad a_{st} = (-1)^t a_{ts}; \qquad \text{and} \qquad a_{st} = -(-1)^t a_{ts}.$$

That is, these are conditions both necessary and sufficient for the existence of the defined symmetry.

By the aid respectively of the first and the second conditions in equations (7) we may transform equation (1) into the following:

$$\sum [a_{ts}(x^t y^s + (-1)^t x^s y^t)] = 0, \qquad t \geqq s;$$

$$\sum [a_{ts}(x^t y^s - (-1)^t x^s y^t)] = 0, \qquad t \geqq s.$$

The condition $t \geqq s$ may evidently be replaced by $t = s,\ s+1, \cdots, n$. Putting these results with the preceding ones we have the theorem:

Excluding curves consisting of isolated points only, the only loci of the nth order possessing fourfold symmetry with respect to a point taken as the origin are those whose rectangular Cartesian equations are of one of the forms

$$\sum [a^{ts}(x^t y^s + (-1)^t x^s y^t)] = 0,$$

$$\sum [a_{ts}(x^t y^s - (-1)^t x^s y^t)] = 0,$$

where n must be even and

$$s = 0,\ 1,\ 2, \cdots, n,$$

$$t = s,\ s+1, \cdots, n;$$

subject to the condition that

$$t + s = 0,\ 2,\ 4, \cdots, n.$$

Alabama Presbyterian College,
Anniston, Ala.,
June, 1907.